Random Convex Approximations of Ambiguous Chance Constrained Programs

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joint work with Eilyan Bitar and Kevin Tang

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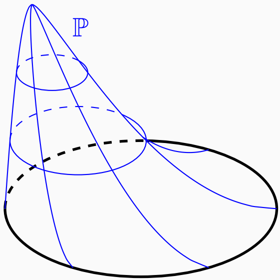
School of Electrical and Computer Engineering, Cornell University

Chance Constrained Program (CCP)



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$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & x \in \mathcal{X} \\ & \mathbb{P}\{f(x,\delta) \leq 0\} \geq 1 - \epsilon. \end{aligned}$$

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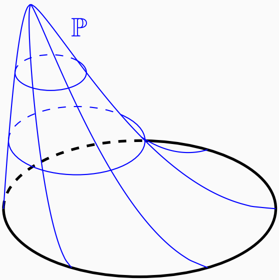

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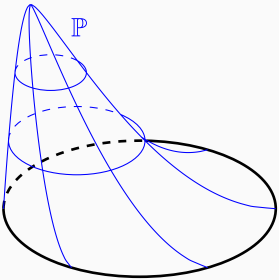

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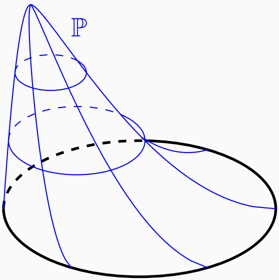

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- $\epsilon \in [0,1]$ is the acceptable constraint violation probability.

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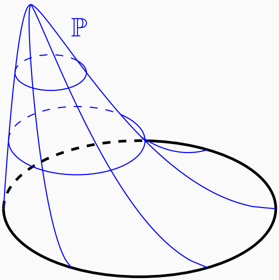

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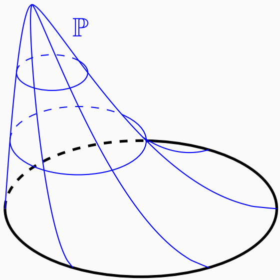

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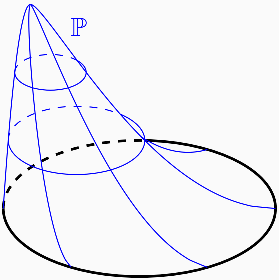

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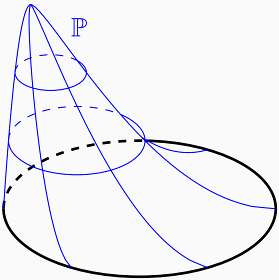

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- The *uncertainty set* Δ;
- $\mathcal{X}$ is closed and convex;
- $f : \mathcal{X} \times \Delta \to \mathbb{R}$ is closed and convex in x for each $\delta \in \Delta$.

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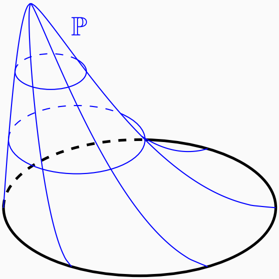

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Assumptions:

- δ is a random variable distributed over Δ according to $\mathbb{P}$.

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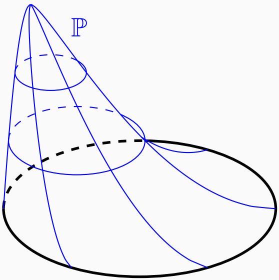

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Shortcoming:

- Non-convexity of the feasible region $\Rightarrow$ hard to solve.

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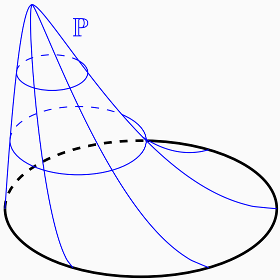

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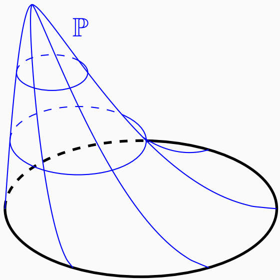
Shortcoming:

- Non-convexity of the feasible region $\Rightarrow$ hard to solve.
- Convex inner approximation of the feasible region for some special cases (Nemirovski and Shapiro, 2006).

A. Nemirovski and A. Shapiro, "Convex Approximations of Chance Constrained Programs," 2006.

Approximation via Sampling



Question: How to approximate CCP?

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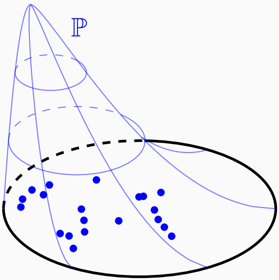

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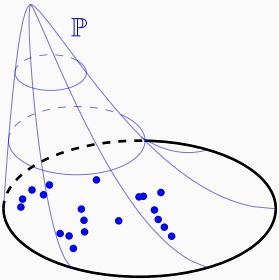
- Suppose we are able to procure N IID samples

$$\delta_1, \ldots, \delta_N \sim \mathbb{P}$$

from $\mathbb{P}$. How can we use these samples to approximate CCP?

Approximation via Sampling



Question: How to approximate CCP with the IID samples?

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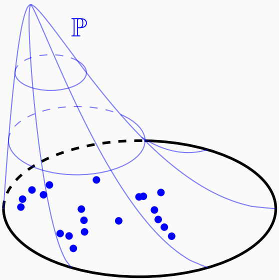
- The basic idea is to replace the chance constraint with other constraints.

Sample Average Approximation (SAA)



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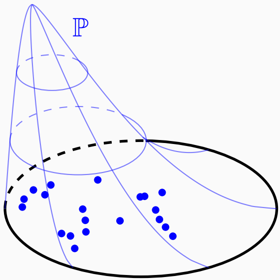
- E.g., using *sample average approximation (SAA)*

$$\frac{1}{N}\sum_{i=1}^{N} \mathbf{1}\{f(x,\delta_i) \leq 0\} \geq 1 - \epsilon$$

gives a mixed integer program (Ahmed and Shapiro, 2008).

S. Ahmed and A. Shapiro, "Solving Chance-Constrained Stochastic Programs via Sampling and Integer Programming," 2008.

Sampled Convex Program (SCP)



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- Another way is to enforce the "sampled" constraints

$$f(x,\delta_i) \le 0, \quad i = 1, \ldots, N,$$

which results in a *sampled convex program (SCP)*.

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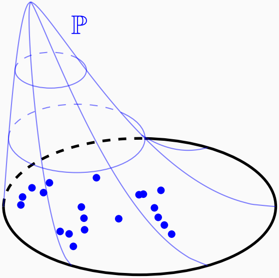

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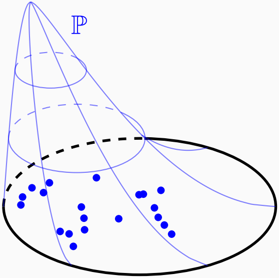

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- The computational complexity is decided by f.

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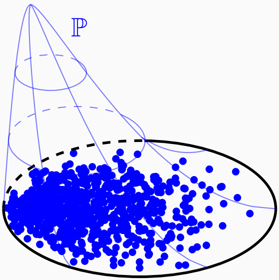

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Properties:

- The computational complexity is decided by f.
- Let x_N^{0*} be the optimal solution to the SCP.

$$\lim_{N \to \infty} \mathbb{P}\left\{f(x_N^{0*}, \delta) \leq 0\right\} \to 1.$$

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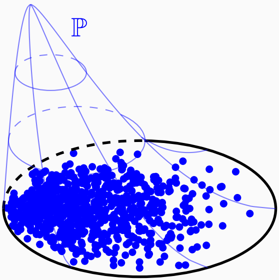

Question: How large N should be s.t. x_N^{0*} is feasible to CCP with high probability?

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Violation Probability

- Let the feasible set of CCP be

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Question: How many samples are needed so that $x_N^{0*} \in \mathcal{X}_\epsilon^0$ with probability at least $1 - \beta$?

$$\mathbb{P}^N\left\{x_N^{0*} \in \mathcal{X}_\epsilon^0\right\} \geq 1 - \beta.$$

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- How to bound the *violation probability* by β?

$$\mathbb{P}^N\left\{x_N^{0*} \notin \mathcal{X}_\epsilon^0\right\} \le \beta.$$

Sampled Convex Program (SCP)

Theorem (Campi and Garatti, 2008; Calafiore, 2010)

$$\mathbb{P}^N \left\{ x_N^{0*} \notin \mathcal{X}_\epsilon^0 \right\} \leq \Phi(\epsilon),$$

where

$$\Phi(\epsilon) := \begin{cases} 1, & \epsilon \in (-\infty, 0], \\ \sum_{i=1}^{n-1} \binom{N}{i} \epsilon^i (1-\epsilon)^{N-i}, & \epsilon \in (0, 1], \\ 0, & \epsilon \in (1, \infty). \end{cases}$$

M. C. Campi and S. Garatti, "The Exact Feasibility of Randomized Solutions of Uncertain Convex Programs," 2008.
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Theorem (Campi and Garatti, 2008; Calafiore, 2010)

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- We can define the *sample size requirement*

$$N(\epsilon, \beta) := \min \left\{ N \in \mathbb{N} : \Phi(\epsilon) \leq \beta \right\}.$$

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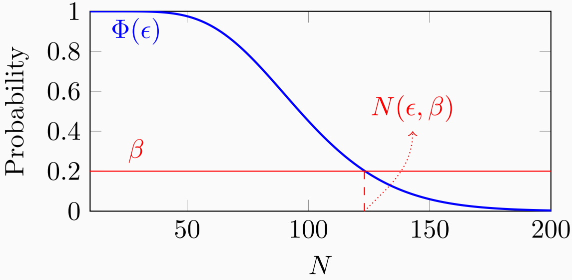

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Corollary (Campi and Garatti, 2008; Calafiore, 2010)

$$N(\epsilon, \beta) \leq \frac{2}{\epsilon} \left(\ln \frac{1}{\beta} + n \right)$$

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- Notice that the results hold for "any" distribution $\mathbb{P}$.

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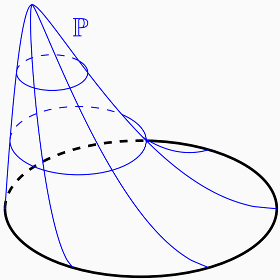
Sampling a Misspecified Model



Issue: In practice, one might have limited access to IID samples from $\mathbb{P}$.

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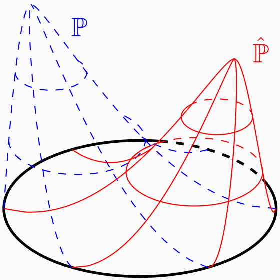

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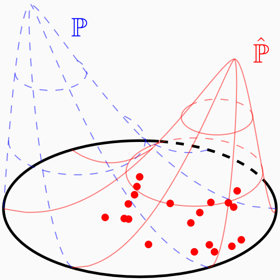
- Sampling efficiently from a (misspecified) model $\hat{\mathbb{P}} \neq \mathbb{P}$.



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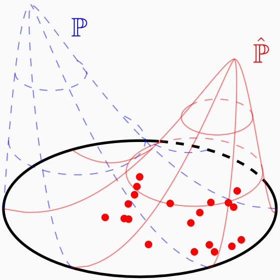

Issue: In practice, one might have limited access to IID samples from $\mathbb{P}$.

Question: How "misspecified" is the model $\hat{\mathbb{P}}$?

(How *ambiguous* is our information of $\mathbb{P}$?)

- Sampling efficiently from a (misspecified) model $\hat{\mathbb{P}} \neq \mathbb{P}$.

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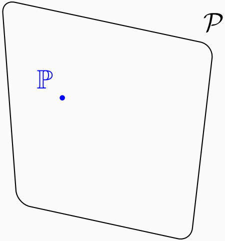
Characterizing Distributional Ambiguity



Approach: Let the *ambiguity set* $\mathcal{P}$ be the set where $\mathbb{P}$ lies in.

Question: How to specify $\mathcal{P}$?

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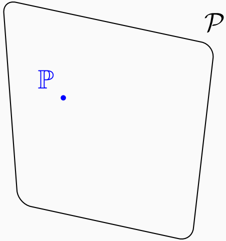

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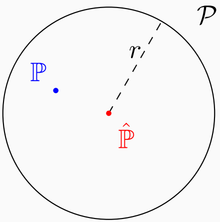
- Moment based specifications (e.g., mean and variance) (Calafiore and El Ghaoui, 2006).

G. C. Calafiore and L. El Ghaoui, "On Distributionally Robust Chance-Constrained Linear Programs," 2006.



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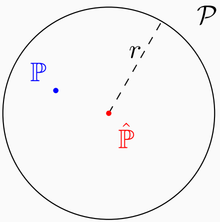
- Alternatively, we define the $\mathcal{P}$ to be

$$\rho(\mathbb{P}, \hat{\mathbb{P}}) \leq r,$$

where ρ is a distance/metric over probability measures on Δ.



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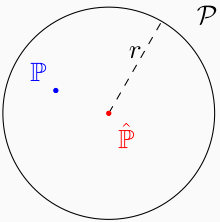

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Approach: Enforce the chance constraint for every single elements in $\mathcal{P}$.

Ambiguous chance constrained program (ACCP):

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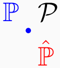
Ambiguous Chance Constrained Program (ACCP)



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- When $r = 0$, we recover the ambiguity-free formulation.

Ambiguous chance constrained program (ACCP):

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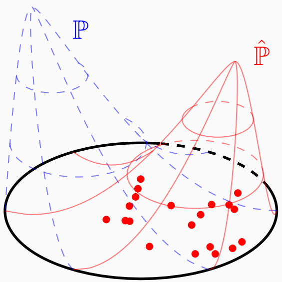
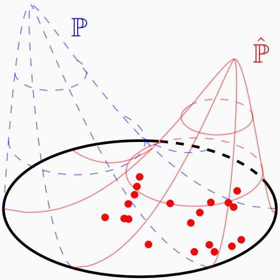


Question: How to approximate ACCP with the IID samples from $\hat{\mathbb{P}}$?

Approximating ACCP via Sampling



Question: How to approximate ACCP with the IID samples from $\hat{\mathbb{P}}$?

Idea: For CCP, we have SCP.

Sampled convex program (SCP):

$$
\begin{aligned}
\text{minimize} \quad & c^\top x \\
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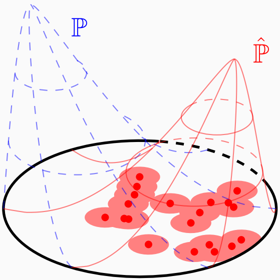
\end{aligned}
$$

Robust Sampled Convex Program (RSCP)



Question: How to approximate ACCP with the IID samples from $\hat{\mathbb{P}}$?

Approach: When ρ is the Prokhorov metric, *robust sampled convex program (RSCP)* can approximate ACCP (Erdoğan and Iyengar, 2006).

Robust sampled convex program (RSCP):

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & x \in \mathcal{X} \\ & f(x, z) \leq 0, \quad \forall\, z \in \bigcup_{i=1}^{N} B_r(\hat{\delta}_i) \cap \Delta. \end{aligned}$$

Prokhorov Metric

Definition

Given two probability measures $\mathbb{P}, \mathbb{Q} \in \mathcal{M}(\Delta)$, the Prokhorov metric is defined as

$$\rho_p(\mathbb{P}, \mathbb{Q}) := \inf\{\gamma > 0 : \mathbb{P}\{A\} \leq \mathbb{Q}\{A^\gamma\} + \gamma, \ \forall\ A \in \mathcal{B}(\Delta)\},$$

where $A^\gamma := \{y \in \Delta : \inf_{z \in A} \|y - z\| < \gamma\}$ denotes the γ-neighborhood of the set A. Here, $\| \cdot \|$ is a suitable norm on the space Δ.

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- Evaluating Prokhorov metric is not trivial.
- However, it can be related to other metrics through inequalities (Gibbs and Su, 2002).

Violation Probability

- Similarly, we can define the feasible set of ACCP

$$\mathcal{X}_\epsilon^r := \left\{ x \in \mathcal{X} : \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{P}\left\{ f(x, \delta) \leq 0 \right\} \geq 1 - \epsilon \right\}.$$

- Let the optimal solution to RSCP be x_N^{r*}, the violation probability should be bounded by β

$$\hat{\mathbb{P}}^N \left\{ x_N^{r*} \notin \mathcal{X}_\epsilon^r \right\} \leq \beta.$$

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- Can we find an upper bound on the violation probability?

Robust Sampled Convex Program (RSCP)

Theorem (Erdoğan and Iyengar, 2006)

$$\hat{\mathbb{P}}^N\left\{x_N^{r*} \notin \mathcal{X}_\epsilon^r\right\} \leq \left(\frac{eN}{n}\right)^n e^{-(\epsilon-r)(N-n)}.$$

- The sample size requirement:

$$\overline{N}(\epsilon - r, \beta) := \min\left\{N \in \mathbb{N} : \left(\frac{eN}{n}\right)^n e^{-(\epsilon-r)(N-n)} \leq \beta\right\}.$$

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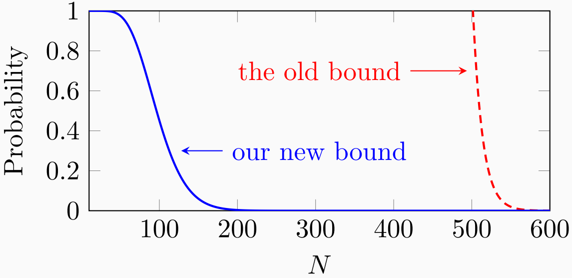
Theorem (Tseng, Bitar and Tang, 2016)

$$\hat{\mathbb{P}}^N \left\{x_N^{r*} \notin \mathcal{X}_\epsilon^r\right\} \leq \Phi(\epsilon - r)$$

- The sample size requirement for our new bound is $N(\epsilon - r, \beta)$.

Tighter Bound for RSCP Approximation to ACCP under ρ_p

- Our new bound on the violation probability improves upon the existing bound,



Tighter Bound for RSCP Approximation to ACCP under ρ_p

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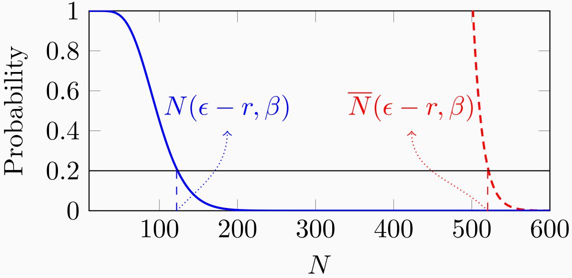

which implies smaller sample size requirement, i.e.,

$$N(\epsilon - r, \beta) \leq \overline{N}(\epsilon - r, \beta).$$

Tighter Bound for RSCP Approximation to ACCP under ρ_p

- Fixing $n = 10$, $r = 0.1$ and $\beta = 10^{-5}$, we compare the sample size requirement implied by our new bound $N(\epsilon - r, \beta)$ and the old bound $\overline{N}(\epsilon - r, \beta)$ under different ϵ.

ϵ	0.15	0.125	0.11	0.105	0.1025	0.101
$N(\epsilon - r, \beta)$	581	1171	2942	5895	11799	29513
$\overline{N}(\epsilon - r, \beta)$	1434	3175	8960	19460	41986	115027

The Idea

- By defining

$$g(x, \hat{\delta}) = \sup_{z \in B_r(\hat{\delta}) \cap \Delta} f(x, z),$$

we can transform RSCP to be in the form of SCP

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & x \in \mathcal{X} \\ & f(x, z) \le 0, \quad \forall\, z \in \bigcup_{i=1}^{N} B_r(\hat{\delta}_i) \cap \Delta. \end{aligned}$$

Optimal solution: x_N^{r*}; ACCP feasible set: $\mathcal{X}_{(\cdot)}^r$.

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$$g(x, \hat{\delta}) = \sup_{z \in B_r(\hat{\delta}) \cap \Delta} f(x, z),$$

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$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & x \in \mathcal{X} \\ & g(x, \hat{\delta}_i) \leq 0, \quad i = 1, \ldots, N. \end{aligned}$$

Optimal solution: $y_N^{0*} = x_N^{r*}$; CCP feasible set: $\mathcal{Y}^0_{(\cdot)}$.

The Idea

- By defining

$$g(x, \hat{\delta}) = \sup_{z \in B_r(\hat{\delta}) \cap \Delta} f(x, z),$$

  we can transform RSCP to be in the form of SCP.

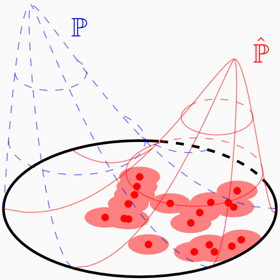
- By the definition of Prokhorov metric,

$$x_N^{r*} \notin \mathcal{X}_\epsilon^r \quad \text{implies} \quad y_N^{0*} \notin \mathcal{Y}_{\epsilon-r}^0.$$

- Therefore

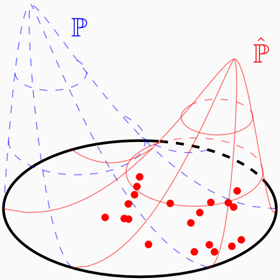
$$\hat{\mathbb{P}}^N \left\{ x_N^{r*} \notin \mathcal{X}_\epsilon^r \right\} \leq \hat{\mathbb{P}}^N \left\{ y_N^{0*} \notin \mathcal{Y}_{\epsilon-r}^0 \right\} \leq \Phi(\epsilon - r).$$

Approximating ACCP via Sampling



Question: Do we really need to use RSCP?

Approximating ACCP via Sampling



Question: Do we really need to use RSCP? Can SCP approximate ACCP with performance guarantee?

Sampled convex program (SCP):

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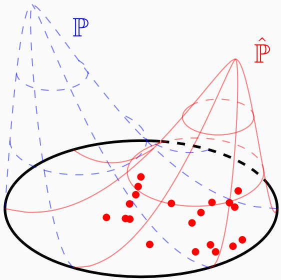
Approximating ACCP via Sampling



Question: Do we really need to use RSCP? Can SCP approximate ACCP with performance guarantee?

Answer: Yes, we can approximate ACCP by SCP.

Sampled convex program (SCP):

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & x \in \mathcal{X} \\ & f(x, \hat{\delta}_i) \leq 0, \quad i = 1, \dots, N. \end{aligned}$$

Perturbed Risk Level

- The key idea is the *perturbed risk level*.

Definition

The *perturbed risk level* $\nu_\epsilon^r \in [0,1]$ associated with the ambiguity set $\mathcal{P}$ is defined as

$$\nu_\epsilon^r := \sup\{\alpha : \hat{\mathbb{P}}\{A\} \le \alpha \Rightarrow \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\{A\} \le \epsilon,\ \forall A \in \mathcal{B}(\Delta)\},$$

where $\mathcal{B}(\Delta)$ is the Borel σ-algebra on Δ. We define $\nu_\epsilon^r = 0$ if the supremum does not exist.

The Idea

- From the definition of the perturbed risk level,

$$x_N^{0*} \notin \mathcal{X}_\epsilon^r \quad \text{implies} \quad x_N^{0*} \notin \mathcal{X}_{\nu_\epsilon^r}^0.$$

- We know the violation probability bound for SCP

$$\hat{\mathbb{P}}^N \left\{ x_N^{0*} \notin \mathcal{X}_{\nu_\epsilon^r}^0 \right\} \leq \Phi(\nu_\epsilon^r).$$

- Therefore

$$\hat{\mathbb{P}}^N \left\{ x_N^{0*} \notin \mathcal{X}_\epsilon^r \right\} \leq \hat{\mathbb{P}}^N \left\{ x_N^{0*} \notin \mathcal{X}_{\nu_\epsilon^r}^0 \right\} \leq \Phi(\nu_\epsilon^r).$$

SCP Approximation to ACCP

Lemma (Tseng, Bitar and Tang, 2016)

$$\hat{\mathbb{P}}^N \left\{ x_N^{0*} \notin \mathcal{X}_\epsilon^r \right\} \leq \Phi(\nu_\epsilon^r).$$

Moreover, it holds that $\Phi(\nu_\epsilon^r) \leq \Phi(\nu)$ *for all* $\nu \leq \nu_\epsilon^r$*.*

Lemma (Tseng, Bitar and Tang, 2016)

$$\hat{\mathbb{P}}^N \left\{ x_N^{0*} \notin \mathcal{X}_\epsilon^r \right\} \leq \Phi(\nu_\epsilon^r).$$

Moreover, it holds that $\Phi(\nu_\epsilon^r) \leq \Phi(\nu)$ *for all* $\nu \leq \nu_\epsilon^r$*.*

- As such, a lower bound on ν_ϵ^r leads to an upper bound on the violation probability.

- We then derive the lower bounds for some probability metrics.

Proposition (Tseng, Bitar and Tang, 2016)

Fix $\epsilon \in [0,1]$ *and* $r \geq 0$. *For each of the following distance functions, the corresponding perturbed risk level* ν_ϵ^r *satisfies the lower bound:*

(a) *Total variation metric,* ρ_{tv}: $\nu_\epsilon^r \geq \epsilon - r$.

(b) *Hellinger metric,* ρ_h: $\nu_\epsilon^r \geq \max\left(\sqrt{\epsilon} - r,\ 0\right)^2$.

(c) *Relative entropy distance,* ρ_e: $\nu_\epsilon^r \geq \sup_{\lambda > 0} \frac{e^{-r}(\lambda+1)^\epsilon - 1}{\lambda}$.

(d) χ^2*-distance,* ρ_{χ^2}: $\nu_\epsilon^r \geq \epsilon + \frac{r}{2} - \sqrt{r\epsilon + \frac{r^2}{4}}$.

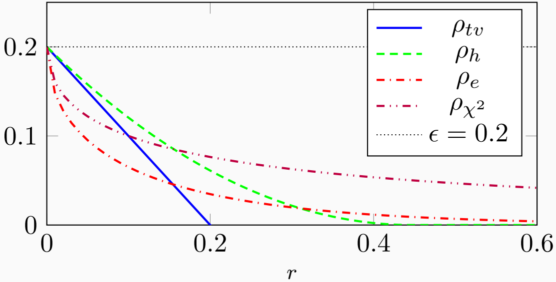


Figure 1: Plot of lower bound on the perturbed risk level ν_ϵ^r versus r for $\epsilon = 0.2$. Each curve corresponds to a different distance function.

Sample Size Requirement using SCP

- Fixing $n = 10$, $r = 0.1$ and $\beta = 10^{-5}$, we compare the sample size requirement implied by the total variation metric (N_{tv}), Hellinger metric (N_h), relative entropy distance (N_e), and χ^2-distance (N_{χ^2}). Let $N_0 = N(\epsilon, \beta)$.

ϵ	0.2	0.15	0.125	0.11	0.105	0.1025	0.101
N_{tv}	285	581	1171	2942	5895	11799	29513
N_h	235	348	449	540	578	599	612
N_e	444	762	1098	1438	1591	1678	1734
N_{χ^2}	285	426	552	664	711	736	752
N_0	137	187	226	258	271	278	282

Summary

Problem	Method	Violation Probability Bound
CCP	SCP	$\mathbb{P}^N \left\{ x_N^{0*} \notin \mathcal{X}_\epsilon^0 \right\} \leq \Phi(\epsilon)$
ACCP (Prokhorov)	RSCP	$\hat{\mathbb{P}}^N \left\{ x_N^{r*} \notin \mathcal{X}_\epsilon^r \right\} \leq \Phi(\epsilon - r)$
ACCP	SCP	$\hat{\mathbb{P}}^N \left\{ x_N^{0*} \notin \mathcal{X}_\epsilon^r \right\} \leq \Phi(\nu_\epsilon^r)$

Summary

Problem	Method	Sample Complexity
CCP	SCP	$N(\epsilon, \beta) \leq \frac{2}{\epsilon}\left(\ln \frac{1}{\beta} + n\right)$
ACCP (Prokhorov)	RSCP	$N(\epsilon - r, \beta) \leq \frac{2}{\epsilon - r}\left(\ln \frac{1}{\beta} + n\right)$
ACCP	SCP	$N(\nu_\epsilon^r, \beta) \leq \frac{2}{\nu_\epsilon^r}\left(\ln \frac{1}{\beta} + n\right)$

Conclusion

- We improve the existing bound on the violation probability of RSCP approximation to ACCP under Prokhorov metric. The new bound recovers the ambiguity-free bound when the radius of the ambiguity set is zero.
- Our results serve as tools for data-driven optimization. When limited IID samples from the true distribution is available, our results allow one to generate IID samples from (potentially) misspecified model ("No model is perfect") with bounds on the violation probability and the sample complexity.

Future Directions

- Construction of ambiguity set $\mathcal{P}$ using limited samples from $\mathbb{P}$.
- Techniques for parallelization.
- Probabilistic bounds on the optimality gap.
- Generalizing to non-convex f (e.g., indefinite quadratic).

Questions & Answers

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